

Lidstone Fractal Interpolation and Error Analysis

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Abstract

In the present paper, the notion of Lidstone Fractal Interpolation Function (*Lidstone FIF*) is introduced to interpolate and approximate data generating functions that arise from real life objects and outcomes of several scientific experiments. A Lidstone FIF extends the classical Lidstone Interpolation Function which is generally found not to be satisfactory in interpolation and approximation of such functions. For a data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ with $N, p \in \mathbb{N}$, the existence of Lidstone FIF is proved in the present work and a computational method for its construction is developed. The constructed Lidstone FIF is a $C^{2p}[x_0, x_N]$ fractal function ℓ_α satisfying $\ell_\alpha^{(2k)}(x_n) = y_{n,2k}$, $n = 0, 1, \dots, N$, $k = 0, 1, \dots, p$. Our error estimates establish that the order of L^∞ -error in approximation of a data generating function in $C^{2p}[x_0, x_N]$ by Lidstone FIF is of the order N^{-2p} , while L^∞ -error in approximation of $2k$ -order derivative of the data generating function by corresponding order derivative of Lidstone FIF is of the order $N^{-(2p-2k)}$. The results found in the present work are illustrated for computational constructions of a Lidstone FIF and its derivatives with an example of a data generating function.

Key Words : Lidstone, Interpolation, Approximation, Fractal, FIF, IFS, Error Estimate

1 Introduction

The Lidstone Polynomials [13] were introduced to offer an approximation of sufficient number of times continuously differentiable functions at two points instead of their one point approximation given by Taylor Polynomials. A classical Lidstone Interpolation Function on the interval $[x_0, x_N]$ is comprised of a Lidstone Polynomial for each subinterval arising from a partition of the interval. It approximates a function in $C^{2p}[x_0, x_N]$, the class of $2p$ times continuously differentiable functions, such that the Lidstone Interpolation Function and all its even order derivatives up to the order $2p$ agree with the given function and its corresponding derivatives at finitely many abscissas of data points in the interval including its end points. Such an interpolation is widely used in the boundary value problem consisting of the $2p^{th}$ -order ordinary differential equation $y^{(2p)} = -f(x, y, y', \dots, y^{(2p-1)})$ with Lidstone Boundary Conditions $y^{(2i)}(x_0) = \alpha_i$, $y^{(2i)}(x_N) = \beta_i$, $i = 0, 1, \dots, p$ [1, 3, 8, 16]. However the classical Lidstone Interpolation is generally not suited for approximation of fractal functions [5]. Such functions quite often arise in various science and engineering problems [4, 9, 11, 12], medicine[10], economics [15, 17], arts[7], and music[14]. In the present paper we introduce the notion of Lidstone Fractal Interpolation to approximate such fractal functions with respect to L^∞ -norm.

The organization of the paper is as follows. In section 2, we review some basic definitions and results on classical Lidstone Interpolation that are used in the later sections of the paper. In Section 3, for an interpolation data in the Euclidean plane \mathbb{R}^2 , the notion of Lidstone FIF is introduced, its existence is established and a computational method of construction of Lidstone FIF is developed. The convergence of Lidstone FIF and its even order derivatives to the data generating function and its corresponding derivatives are studied in Section 4. To this end, first the continuous dependence of Lidstone FIF ℓ_α and its even order derivatives on the parameter α is established. This is followed by proving that, for the classical Lidstone Interpolation Function $\phi = \phi^{(0)}$ and its even order derivatives $\phi^{(2k)}$ and Lidstone FIF $\ell_\alpha = \ell_\alpha^{(0)}$ and its corresponding derivatives $\ell_\alpha^{(2k)}$, the L^∞ -norm $\|\phi^{(2k)} - \ell_\alpha^{(2k)}\|_\infty \rightarrow 0$, $k = 0, 1, 2, \dots, p$, as norm of the partition of the interval $[x_0, x_N]$ consisting abscissas of data points tends to zero. Using these results, it is found in this section that L^∞ -error in approximation of data generating function in $C^{2p}[x_0, x_N]$ by Lidstone FIF is of the order N^{-2p} , while L^∞ -error in approximation of $2k$ -order derivative of data

generating function in $C^{2p}[x_0, x_N]$ by corresponding order derivative of Lidstone FIF is of the order $N^{-(2p-2k)}$. Finally, the results of Section 3 on computational constructions of a Lidstone FIF and its derivatives are illustrated with an example in Section 5 .

2 Classical Lidstone and Piecewise-Lidstone

Interpolation: Basic Concepts

It is known [18] that a function $g : \mathbb{R} \rightarrow \mathbb{R}$ possessing a sufficient number of derivatives can be expanded as Lidstone Series

$$g(x) = \sum_{l=0}^p \left[g^{(2l)}(0)\Lambda_l(1-x) + g^{(2l)}(1)\Lambda_l(x) \right] + R_{p+1}(g, x),$$

where, \mathbb{R} is the real line, $\Lambda_l(x)$ are the *Lidstone Polynomials* defined by means of recursive relations:

$$\left. \begin{aligned} \Lambda_0(x) &= x \\ \Lambda_l''(x) &= \Lambda_{l-1}(x), \quad l \geq 1 \\ \Lambda_l(0) &= \Lambda_l(1) = 0, \quad l \geq 1 \end{aligned} \right\} \quad (2.1)$$

and $R_{p+1}(g, x) = \int_0^1 G_{p+1}(x, t) g^{(2p+2)}(t) dt$, where

$$G_1(x, t) = \begin{cases} (x-1)t, & t \leq x \\ (t-1)x, & x \leq t \end{cases}$$

$$G_p(x, t) = \int_0^1 g_1(x, s) G_{p-1}(s, t) ds, \quad p \geq 2.$$

The following proposition gives an explicit form of Lidstone Polynomial and its bound in the interval $[0, 1]$:

Proposition 2.1 [2] *The Lidstone Polynomial $\Lambda_l(x)$, in the interval $[0, 1]$, can be expressed as*

$$\Lambda_l(x) = (-1)^l \frac{2}{\pi^{2l+1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2l+1}} \sin n\pi x, \quad l \geq 1$$

so that,

$$|\Lambda_l(x)| \leq \frac{1}{3\pi^{2l-1}}; \quad 0 \leq x \leq 1. \quad (2.2)$$

The truncated Lidstone series

$$\sum_{l=0}^p [\alpha_{2l}\Lambda_l(1-x) + \beta_{2l}\Lambda_l(x)]$$

for $\alpha_{2l}, \beta_{2l} \in \mathbb{R}$, $l = 0, 1, \dots, p$, is known as *Lidstone Interpolating Polynomial* in the interval $[0, 1]$.

To extend Lidstone Interpolating Polynomial to the interval $[x_0, x_N]$, we have

Definition 2.1 *A real polynomial $q(x)$ of degree $2p + 1$, satisfying the Lidstone Conditions $q^{(2l)}(x_0) = y_{0,2l}$, $q^{(2l)}(x_N) = y_{N,2l}$, $0 \leq l \leq p$, $y_{0,2l}, y_{N,2l} \in \mathbb{R}$, is known as a Lidstone Interpolating Polynomial in $[x_0, x_N]$.*

For given $y_{0,2l}, y_{N,2l}$, $l = 0, 1, \dots, p$, a representation of the Lidstone Interpolating Polynomial $q(x)$ is given by the following theorem:

Theorem 2.1 [2] *The Lidstone Interpolating Polynomial $q(x)$ can be expressed as*

$$q(x) = \sum_{l=0}^p \left[y_{0,2l} \Lambda_l \left(\frac{x_N - x}{x_N - x_0} \right) + y_{N,2l} \Lambda_l \left(\frac{x - x_0}{x_N - x_0} \right) \right] (x_N - x_0)^{2l}.$$

It is apparent that Lidstone Interpolating Polynomial interpolates a data given only at two points. To interpolate a data given at more than two points, Lidstone Interpolation Functions defined below are employed. Let $\tilde{S}^p[x_0, x_N]$, where $-\infty < x_0 < x_N < \infty$ and p is a positive integer, be the set of all real-valued functions $g(x)$ that satisfy the following conditions:

- (i) The function g is p times continuously differentiable on $[x_0, x_N]$.
- (ii) There exists a partition $\Delta : x_0 < x_1 < \dots < x_N$ such that on each open subinterval (x_{n-1}, x_n) , the p^{th} -derivative $g^{(p)}$ is continuously differentiable.

(iii) The sup-norm of $g^{(p+1)}$ is finite, i.e.,

$$\|g^{(p+1)}\|_\infty = \max_{1 \leq i \leq N} \sup_{x \in (x_{n-1}, x_n)} |g^{(p)}(x)| < \infty.$$

For a fixed uniform partition $\Delta : x_0 < x_1 < \dots < x_N$ of the interval $[x_0, x_N]$, define

$$L_{p+1}^\Delta = \{\varphi \mid \varphi \in C[x_0, x_N] \text{ and } \varphi \text{ is a polynomial of degree at most } 2p+1 \text{ in each sub interval } [x_{n-1}, x_n]\}$$

where, $C[x_0, x_N]$ is the set of all continuous functions in $[x_0, x_N]$. It is easily seen that L_{p+1}^Δ is a linear space with usual point-wise addition and scalar multiplication of functions.

Definition 2.2 For a given function $g \in C^{2p}[x_0, x_N]$, $L_{p+1}^\Delta g(x)$ is called the L_{p+1}^Δ -interpolate (also called Lidstone Interpolation Function) of $g(x)$, if $L_{p+1}^\Delta g \in L_{p+1}^\Delta$ and $\frac{d^{2k}}{dx^{2k}} L_{p+1}^\Delta g(x_n) = g^{(2k)}(x_n); 0 \leq k \leq p, 0 \leq n \leq N$.

In view of Theorem 2.1, L_{p+1}^Δ -interpolate for a function $g \in C^{2p}[x_0, x_N]$ uniquely exists and, for $x \in [x_{n-1}, x_n]$,

$$L_{p+1}^\Delta g(x) = \sum_{l=0}^p \left[g^{(2l)}(x_{n-1}) \Lambda_l \left(\frac{x_n - x}{x_n - x_{n-1}} \right) + g^{(2l)}(x_n) \Lambda_l \left(\frac{x - x_{n-1}}{x_n - x_{n-1}} \right) \right] (x_n - x_{n-1})^{2l}.$$

Thus,

$$L_{p+1}^\Delta g(x) = \sum_{n=0}^N \sum_{l=0}^p r_{p,n,l}(x) g^{(2l)}(x_n)$$

where,

$$r_{p,n,l}(x) = \begin{cases} \Lambda_l \left(\frac{x - x_{n-1}}{x_n - x_{n-1}} \right) (x_n - x_{n-1})^{2l}, & x_{n-1} \leq x \leq x_n, \quad 1 \leq n \leq N \\ \Lambda_l \left(\frac{x_{n+1} - x}{x_{n+1} - x_n} \right) (x_{n+1} - x_n)^{2l}, & x_n \leq x \leq x_{n+1}, \quad 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that the set of functions $\{r_{p,n,l}; n = 0, 1, \dots, N, l = 0, 1, \dots, p\}$ forms a basis of the linear space L_{p+1}^Δ .

The error in approximation of a function belonging to $\tilde{S}^{2p}[x_0, x_N]$ and its even order derivatives by an L_{p+1}^Δ -interpolate and its corresponding derivatives is given by the following theorem:

Theorem 2.2 [2] *Let $g \in \tilde{S}^{2p}[x_0, x_N]$. Then*

$$\left\| \frac{d^k}{dx^k} (g - L_{p+1}^\Delta g) \right\|_\infty \leq 2d_{2p,k} \|g^{(2p)}\|_\infty \|\Delta\|^{2p-k}, \quad 0 \leq k \leq 2p \quad (2.3)$$

where, $\|\Delta\| = \max\{x_n - x_{n-1} ; \quad 1 \leq n \leq N\}$ and

$$d_{2p,k} = \begin{cases} \frac{(-1)^{p-i} E_{2p-2i}}{2^{2p-2i} (2p-2i)!}, & k = 2i, \quad 0 \leq i \leq p \\ (-1)^{p-i+1} \frac{2(2^{2p-2i}-1)}{(2p-2i)!} B_{2p-2i}, & k = 2i+1, \quad 0 \leq i \leq p-1 \\ 2, & k = 2p+1 \end{cases} \quad (2.4)$$

E_{2p} and B_{2p} being $2p^{th}$ Euler and Bernoulli numbers respectively.

3 Construction of Lidstone FIF

In this section, we introduce Lidstone Fractal Interpolation Function, prove the existence and give a computational method for its construction. We begin with the definition of Lidstone FIF for a given data:

Let $x_0 < x_1 < \dots < x_N$ be a partition of the interval $[x_0, x_N]$ and $\Theta_{2p} = \{\alpha \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{R}^N, |\alpha_n| < a_n^{2p}\}$, where $a_n = \frac{x_n - x_{n-1}}{x_N - x_0}$. For $\alpha \in \Theta_{2p}$, consider the Iterated Function System (IFS) $\{\mathbb{R}^2; w_n(x, y) = (L_n(x), F_n(x, y)), n = 1, 2, \dots, N\}$, where $L_n : \mathbb{R} \rightarrow \mathbb{R}$ and $F_n : \mathbb{R}^2 \rightarrow \mathbb{R}$, for $n = 1, 2, \dots, N$, are given by,

$$\left. \begin{aligned} L_n(x) &= a_n x + b_n \\ F_n(x, y) &= \alpha_n y + q_n(x) \end{aligned} \right\} \quad (3.1)$$

with $b_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}$ and $q_n \in C^{2p}(\mathbb{R})$. For a given set of real numbers y_n , $n = 0, 1, \dots, N$, if the IFS satisfies the join up conditions

$$\left. \begin{aligned} w_n(x_0, y_0) &= (x_{n-1}, y_{n-1}) \\ w_n(x_N, y_N) &= (x_n, y_n) \end{aligned} \right\}$$

for $n = 1, 2, \dots, N$, then the attractor of the IFS is the graph of a C^{2p} -fractal function $\ell_\alpha : [x_0, x_N] \rightarrow \mathbb{R}$ such that $\ell_\alpha(x_n) = y_n$.

Definition 3.1 The C^{2p} -fractal function ℓ_α , generated by the IFS $\{\mathbb{R}^2; w_n(x, y) = (a_n x + b_n, \alpha_n y + q_n(x)), n = 1, 2, \dots, N\}$, is called *Lidstone Fractal Interpolation Function (Lidstone FIF)* for a data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$, if each q_n is a *Lidstone Interpolating Polynomial* of degree at most $2p + 1$ in $[x_0, x_N]$ and $\ell_\alpha^{(2k)}(x_n) = y_{n,2k}$.

The tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{R}^N$ is called *scaling factor* of the Lidstone FIF ℓ_α

The existence and the method of construction of Lidstone FIF is given by the following theorem:

Theorem 3.1 For given data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$, with abscissas as partition points of $[x_0, x_N]$ and $\alpha \in \Theta_{2p}$, the Lidstone FIF ℓ_α uniquely exists.

Proof Consider IFS $\{\mathbb{R}^2; w_n(x, y) = (L_n(x), F_n(x, y)), n = 1, 2, \dots, N\}$, where $L_n : \mathbb{R} \rightarrow \mathbb{R}$, $F_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ are given by (3.1) and the polynomials q_n of degree at most $2p + 1$ are chosen as follows:

Let the maps $F_{n,k}(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$, $k = 0, 1, \dots, p$, be defined by

$$\left. \begin{aligned} F_{n,0}(x, y) &= F_n(x, y) \\ F_{n,2k}(x, y) &= \frac{\alpha_n y + q_n^{(2k)}(x)}{a_n^{2k}}, k = 1, 2, \dots, p, \end{aligned} \right\}.$$

The polynomials $q_n(x)$ are now chosen to satisfy the join up conditions

$$F_{n,2k}(x_0, y_{0,2k}) = y_{n-1,2k} \text{ and } F_{n,2k}(x_N, y_{N,2k}) = y_{n,2k} \text{ for } k = 0, 1, \dots, p$$

so that

$$\frac{\alpha_n y_{0,2k} + q_n^{(2k)}(x_0)}{a_n^{2k}} = y_{n-1,2k} \quad \text{and} \quad \frac{\alpha_n y_{N,2k} + q_n^{(2k)}(x_N)}{a_n^{2k}} = y_{n,2k}$$

for $k = 1, 2, \dots, p$. Consequently, the chosen polynomials satisfy

$$\left. \begin{aligned} q_n^{(2k)}(x_0) &= a_n^{2k} y_{n-1,2k} - \alpha_n y_{0,2k} \\ q_n^{(2k)}(x_N) &= a_n^{2k} y_{n,2k} - \alpha_n y_{N,2k} \end{aligned} \right\},$$

for $k = 1, 2, \dots, p$.

Equivalently, the polynomials q_n are Lidstone Interpolating Polynomials of degree at most $2p + 1$ in $[x_0, x_N]$ given by (*c.f. Theorem 2.1*)

$$q_n(x) = \sum_{l=0}^p \left[q_n^{(2l)}(x_0) \Lambda_l \left(\frac{x_N - x}{x_N - x_0} \right) + q_n^{(2l)}(x_N) \Lambda_l \left(\frac{x - x_0}{x_N - x_0} \right) \right] (x_N - x_0)^{2l}. \quad (3.2)$$

for the two point data $(x_0, a_n^{2k} y_{n-1,2k} - \alpha_n y_{0,2k}), (x_N, a_n^{2k} y_{n,2k} - \alpha_n y_{N,2k})$.

Now, IFS $\{\mathbb{R}^2; w_n(x, y) = (L_n(x), F_n(x, y)) : n = 1, 2, \dots, N\}$, determines a unique FIF $\ell_\alpha \in C^{2p}[x_0, x_N]$, which is the fixed point of Read-Bajraktarević Operator $T_\alpha : \mathcal{F} \rightarrow \mathcal{F}$ given by

$$T_\alpha \phi(x) = F_n(L_n^{-1}(x), \phi \circ L_n^{-1}(x)); \quad x \in [x_{n-1}, x_n] \quad (3.3)$$

where,

$$\mathcal{F} = \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi \text{ is continuous and } \phi(x_0) = y_{0,0}, \phi(x_N) = y_{N,0}\}.$$

Thus, for $x \in [x_{n-1}, x_n]$, $\ell_\alpha(x) = \alpha_n \ell_\alpha \circ L_n^{-1}(x) + q_n \circ L_n^{-1}(x)$.

For establishing that ℓ_α is the Lidstone FIF for the given data, it remains to be shown that $\ell_\alpha^{(2k)}(x_n) = y_{n,2k}$, for $n = 0, 1, \dots, N$ and $k = 1, 2, \dots, p$. Since, by construction, for $n = 1, 2, \dots, N$, $F_{n,2k}(x_0, y_{0,2k}) = y_{n-1,2k}$ and $F_{n,2k}(x_N, y_{N,2k}) = y_{n,2k}$, it follows that, the IFS $\{\mathbb{R}^2; (L_n(x), F_{n,2k}(x, y)), n = 1, 2, \dots, N\}$ determines an FIF $\ell_\alpha^{(2k)}$. In fact, repeated applications of a result of Barnsley and Harrington ([6], Theorem 1), show that $\ell_\alpha^{(2k)}$ is the $2k^{th}$ derivative of FIF ℓ_α . Further, since $\ell_\alpha^{(2k)}$ is the fixed point of Read-Bajraktarević operator $T_\alpha^{(2k)} : \mathcal{F}^{(2k)} \rightarrow \mathcal{F}^{(2k)}$ given by

$$T_\alpha^{(2k)} \phi(x) = F_{n,2k}(L_n^{-1}(x), \phi \circ L_n^{-1}(x)); \quad x \in [x_{n-1}, x_n] \quad (3.4)$$

where,

$$\mathcal{F}^{(2k)} = \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi \text{ is continuous and } \phi(x_0) = y_{0,2k}, \phi(x_N) = y_{N,2k}\},$$

it follows that $\ell_\alpha^{(2k)}(x) = F_{n,2k}(L_n^{-1}(x), \ell_\alpha^{(2k)} \circ L_n^{-1}(x))$. Since $\ell_\alpha^{(2k)} \in \mathcal{F}^{(2k)}$, $\ell_\alpha^{(2k)}(x_0) = y_{0,2k}$ and $\ell_\alpha^{(2k)}(x_N) = y_{N,2k}$. Further, for $n = 1, 2, \dots, N-1$,

$$\begin{aligned} \ell_\alpha^{(2k)}(x_n) &= F_{n,2k}(L_n^{-1}(x_n), \ell_\alpha^{(2k)} \circ L_n^{-1}(x_n)) \\ &= F_{n,2k}(x_N, \ell_\alpha^{(2k)}(x_N)) \\ &= F_{n,2k}(x_N, y_{N,2k}) \\ &= y_{n,2k}. \end{aligned}$$

Thus, ℓ_α is the Lidstone FIF for the given data.

Remark 3.1 *It follows from the proof of Theorem 3.1 that, for $x \in [x_{n-1}, x_n]$, Lidstone FIF ℓ_α is given by*

$$\ell_\alpha(x) = \alpha_n \ell_\alpha \circ L_n^{-1}(x) + q_n \circ L_n^{-1}(x). \quad (3.5)$$

Consequently, $\ell_\alpha \in C^{2p}[x_0, x_N]$ and is a polynomial of degree at most $2p+1$ the interval $[x_{n-1}, x_n]$, if $\alpha_n = 0$. Thus, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) = 0$, the Lidstone FIF $\ell_\alpha \in L_{p+1}^\Delta$ and, for $x \in [x_{n-1}, x_n]$, is given by

$$\ell_\alpha(x) = \sum_{l=0}^p \left[y_{n-1,2l} \Lambda_l \left(\frac{x_n - x}{x_n - x_{n-1}} \right) + y_{n,2l} \Lambda_l \left(\frac{x - x_{n-1}}{x_n - x_{n-1}} \right) \right] (x_n - x_{n-1})^{2l}.$$

4 Convergence of Lidstone FIF and its Derivatives

The convergence of Lidstone FIF and its even order derivatives to the data generating function and its corresponding derivatives are studied in this section. To this end, the continuous dependence of Lidstone FIF ℓ_α and its even order derivatives on the parameter α is proved first. This is followed by proving that, for the classical Lidstone Interpolation Function $\phi = \phi^{(0)}$ and its even order derivatives $\phi^{(2k)}$ and Lidstone FIF $\ell_\alpha = \ell_\alpha^{(0)}$ and its corresponding derivatives $\ell_\alpha^{(2k)}$, the L^∞ -norm $\|\phi^{(2k)} - \ell_\alpha^{(2k)}\|_\infty \rightarrow 0$, $k = 0, 1, 2, \dots, p$, as norm of the partition of the interval $[x_0, x_N]$ consisting

abscissas of data points tends to zero. Using these results, it is found in this section that L^∞ -error in approximation of data generating function in $C^{2p}[x_0, x_N]$ by Lidstone FIF is of the order N^{-2p} , while L^∞ -error in approximation of $2k$ -order derivative of data generating function in $C^{2p}[x_0, x_N]$ by corresponding order derivative of Lidstone FIF is of the order $N^{-(2p-2k)}$.

Let data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ be generated by a function $g(x) \in C^{2p}[x_0, x_N]$ (i.e., $g^{(2k)}(x_n) = y_{n,2k}$) and $\alpha \in \Theta_{2p}$. In the sequel, we use the following notations: $|\alpha|_\infty = \max_n \{|\alpha_n|\}$, $\rho = \max_k \{|y_{0,2k}|, |y_{N,2k}|\}$ and $\|\ell_\alpha\|_\infty = \sup\{|\ell_\alpha(x)| \mid x \in [x_0, x_N]\}$ and denote the polynomial $q_n(x)$ by $q_n(\alpha_n, x)$, to emphasize that it depends on the parameter α_n , as observed in the proof of Theorem 3.1.

We first prove continuous dependence of Lidstone FIF ℓ_α and its even order derivatives on the parameter α with the help of following proposition:

Proposition 4.1 *Let $q_n(\alpha_n, x)$, $n = 1, 2, \dots, N$, be the polynomials constructed in Theorem 3.1. Then, for $k = 0, 1, 2, \dots, p$ and $x \in [x_0, x_N]$,*

$$\left| \frac{\partial^{2k+1}}{\partial \alpha_n \partial x^{2k}} q_n(\alpha_n, x) \right| \leq \frac{2\rho\pi}{3} \sum_{l=0}^{p-k} \left(\frac{x_N - x_0}{\pi} \right)^{2l} \quad (4.1)$$

Proof The representation of $q_n(\alpha_n, x)$ as given by Equation (3.2) is

$$q_n(\alpha_n, x) = \sum_{l=0}^p \left[q_n^{(2l)}(x_0) \Lambda_l \left(\frac{x_N - x}{x_N - x_0} \right) + q_n^{(2l)}(x_N) \Lambda_l \left(\frac{x - x_0}{x_N - x_0} \right) \right] (x_N - x_0)^{2l} \quad (4.2)$$

where, $q_n^{(2l)}(x_0) = a_n^{2l} y_{n-1,2l} - \alpha_n y_{0,2l}$ and $q_n^{(2l)}(x_N) = a_n^{2l} y_{n,2l} - \alpha_n y_{N,2l}$. Differentiating both sides of (4.2) with respect to α_n

$$\frac{\partial}{\partial \alpha_n} q_n(\alpha_n, x) = (-1) \sum_{l=0}^p \left[y_{0,2l} \Lambda_l \left(\frac{x_N - x}{x_N - x_0} \right) + y_{N,2l} \Lambda_l \left(\frac{x - x_0}{x_N - x_0} \right) \right] (x_N - x_0)^{2l}. \quad (4.3)$$

By Equation (4.3) and Inequality (2.2),

$$\left| \frac{\partial}{\partial \alpha_n} q_n(\alpha_n, x) \right| \leq \frac{2\rho\pi}{3} \sum_{l=0}^p \left(\frac{x_N - x_0}{\pi} \right)^{2l} \quad (4.4)$$

for $x \in [x_0, x_N]$. Further, the properties of Lidstone Polynomials (*c.f.* Section 2) and (4.2) give that, for $1 \leq k \leq p$,

$$\frac{\partial^{2k}}{\partial x^{2k}} q_n(\alpha_n, x) = \sum_{l=0}^{p-k} \left[q_n^{(2l+2k)}(x_0) \Lambda_l \left(\frac{x_N - x}{x_N - x_0} \right) + q_n^{(2l+2k)}(x_N) \Lambda_l \left(\frac{x - x_0}{x_N - x_0} \right) \right] (x_N - x_0)^{2l}.$$

Thus, following the method of derivation of Inequality (4.4),

$$\left| \frac{\partial^{2k+1}}{\partial \alpha_n \partial x^{2k}} q_n(\alpha_n, x) \right| \leq \frac{2\rho\pi}{3} \sum_{l=0}^{p-k} \left(\frac{x_N - x_0}{\pi} \right)^{2l} \quad (4.5)$$

for all $x \in [x_0, x_N]$ and $1 \leq k \leq p$. Inequality (4.1) follows from (4.4) and (4.5).

The continuous dependence of Lidstone FIF ℓ_α on the scaling factor α is deduced by the following theorem:

Theorem 4.1 *Let $\ell_{\alpha'}$ and $\ell_{\alpha''}$ be the Lidstone FIFs with respect to scaling factors $\alpha', \alpha'' \in \Theta_{2p}$ respectively, for the data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$. Then,*

$$\|\ell_{\alpha'} - \ell_{\alpha''}\|_\infty \leq \frac{|\alpha' - \alpha''|_\infty}{1 - |\alpha'|_\infty} (\|\ell_{\alpha''}\|_\infty + M_{0,2p}) \quad (4.6)$$

where, $M_{0,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^p \left(\frac{x_N - x_0}{\pi} \right)^{2l}$.

Proof For $x \in [x_{n-1}, x_n]$, it follows by (3.5) and the mean value theorem that

$$\begin{aligned} |\ell_{\alpha'}(x) - \ell_{\alpha''}(x)| &\leq |\alpha'_n| |\ell_{\alpha'}(L_n^{-1}(x)) - \ell_{\alpha''}(L_n^{-1}(x))| \\ &\quad + |\alpha'_n - \alpha''_n| |\ell_{\alpha''}(L_n^{-1}(x))| \\ &\leq |\alpha'_n| |\ell_{\alpha'}(L_n^{-1}(x)) - \ell_{\alpha''}(L_n^{-1}(x))| \\ &\quad + |\alpha'_n - \alpha''_n| |\ell_{\alpha''}(L_n^{-1}(x))| \\ &\quad + |\alpha'_n - \alpha''_n| \left| \frac{\partial}{\partial \alpha_n} q_n(\xi_n, L_n^{-1}(x)) \right| \end{aligned}$$

for some ξ_n lying between α'_n and α''_n . By Proposition 4.1, the above inequality implies

$$\|\ell_{\alpha'} - \ell_{\alpha''}\|_\infty \leq |\alpha'|_\infty \|\ell_{\alpha'} - \ell_{\alpha''}\|_\infty + |\alpha' - \alpha''|_\infty (\|\ell_{\alpha''}\|_\infty + M_{0,2p})$$

which gives (4.6).

An estimate on L^∞ -error between the classical Lidstone Interpolation Function and Lidstone FIF for a given data is now found in the following corollary of Theorem 4.1:

Corollary 4.1 *Let $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ be a given data and $\alpha \in \Theta_{2p}$. Let ℓ_α be the Lidstone FIF and ϕ be the classical Lidstone Interpolation Function for the above data. Then*

$$\|\ell_\alpha - \phi\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} (\|\phi\|_\infty + M_{0,2p}) \quad (4.7)$$

where, $M_{0,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^p \left(\frac{x_N - x_0}{\pi}\right)^{2l}$.

Proof Inequality (4.7) is an immediate consequence of Inequality (4.6) with $\alpha' = \alpha$ and $\alpha'' = 0$, since $\ell_0 = \phi$.

Remark 4.1 *Since $\alpha \in \Theta_{2p}$ in Theorem 4.1 implies that $|\alpha_n| < a_n^{2p}$, the inequality $|\alpha|_\infty \leq (x_N - x_0)^{-2p} \|\Delta\|^{2p}$ holds, where $\|\Delta\| = \max_k |x_k - x_{k-1}|$ is the norm of the partition $x_0 < x_1 < \dots < x_N$ of the interval $[x_0, x_N]$. Therefore, it follows from Corollary 4.1 that the convergence of Lidstone FIF to Classical Lidstone Interpolation Function is of the order $\|\Delta\|^{2p}$.*

Using Corollary 4.1, the order of convergence of Lidstone FIF to the data generating function is found in the following theorem:

Theorem 4.2 *Let $x_0 < x_1 < \dots < x_N$ be a uniform partition of the interval $[x_0, x_N]$ and $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ be a data generated by the function $g \in C^{2p}[x_0, x_N]$. Let ℓ_α be the Lidstone FIF for the data. Then,*

$$\|g - \ell_\alpha\|_\infty = O(N^{-2p}).$$

Proof Let $L_{p+1}^\Delta g$ be the classical Lidstone Interpolation Function for data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$. Inequality (2.3) gives

$$\|L_{p+1}^\Delta g\|_\infty \leq 2d_{2p,0} \|\Delta\|^{2p} \|g^{(2p)}\|_\infty + \|g\|_\infty$$

where, $d_{2p,0} = \frac{(-1)^p E_{2p}}{2^{2p}(2p)!}$, E_{2p} being $2p^{th}$ -Euler number. Therefore, it follows by Corollary 4.1 that

$$\|\ell_\alpha - L_{p+1}^\Delta g\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} (2d_{2p,0} \|\Delta\|^{2p} \|g^{(2p)}\|_\infty + \|g\|_\infty + M_{0,2p}) \quad (4.8)$$

where, $M_{0,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^p \left(\frac{x_N - x_0}{\pi}\right)^{2l}$. Using Inequalities (2.3) and (4.8), the above inequality becomes

$$\|g - \ell_\alpha\|_\infty \leq \frac{1}{1 - |\alpha|_\infty} [2d_{2p,0} \|\Delta\|^{2p} \|g^{(2p)}\|_\infty + |\alpha|_\infty (\|g\|_\infty + M_{0,2p})].$$

Since $|\alpha|_\infty < \frac{1}{N^{2p}} = \frac{\|\Delta\|^{2p}}{(x_N - x_0)^{2p}}$ implies $\frac{1}{1 - |\alpha|_\infty} \leq \frac{N^{2p}}{N^{2p} - 1}$, it follows from the above inequality that

$$\|g - \ell_\alpha\|_\infty \leq \frac{1}{N^{2p} - 1} [2d_{2p,0} (x_N - x_0)^{2p} \|g^{(2p)}\|_\infty + \|g\|_\infty + M_{0,2p}].$$

so that

$$\|g - \ell_\alpha\|_\infty \leq \vartheta_0 N^{-2p}. \quad (4.9)$$

where

$$\vartheta_0 = \frac{1}{1 - N^{-2p}} [2d_{2p,0} (x_N - x_0)^{2p} \|g^{(2p)}\|_\infty + \|g\|_\infty + M_{0,2p}]$$

Since ϑ_0 is bounded for $N > 1$, the result follows from Inequality (4.9).

Our next result on continuous dependence of even order derivatives $\ell_\alpha^{(2k)}$ of Lidstone FIF ℓ_α on the scaling factor α is used to find estimates on L^∞ -error between even order derivatives of the classical Lidstone Interpolation Function and corresponding derivatives of Lidstone FIF for a given data. These estimates are then employed to determine the order of L^∞ -error in approximation of even order derivatives of data generating function by corresponding derivatives of Lidstone FIF.

Theorem 4.3 Let $\ell_{\alpha'}$ and $\ell_{\alpha''}$ be the Lidstone FIFs with scaling factors $\alpha', \alpha'' \in \Theta_{2p}$ respectively, for a data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$. Then, for $k = 1, 2, \dots, p$,

$$\|\ell_{\alpha'}^{(2k)} - \ell_{\alpha''}^{(2k)}\|_{\infty} \leq \frac{|\alpha' - \alpha''|_{\infty}}{\mu^{2k} - |\alpha'|_{\infty}} (\|\ell_{\alpha''}^{(2k)}\|_{\infty} + M_{2k,2p}) \quad (4.10)$$

where, $\mu = \min\{a_n : 1 \leq n \leq N\}$ and $M_{2k,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^{p-k} \left(\frac{x_N - x_0}{\pi}\right)^{2l}$.

Proof Since the $2k^{th}$ derivative $\ell_{\alpha}^{(2k)}$ of the Lidstone FIF ℓ_{α} , is the fixed point of Read-Bajraktarević Operator $T_{\alpha}^{(2k)}$ defined by (3.4), for $x \in [x_{n-1}, x_n]$,

$$\ell_{\alpha}^{(2k)}(x) = \frac{\alpha_n \ell_{\alpha}^{(2k)} \circ L_n^{-1}(x) + q_n^{(2k)} \circ L_n^{-1}x}{a_n^{2k}}.$$

Thus, by mean value theorem, for each $x \in [x_{n-1}, x_n]$,

$$\begin{aligned} |\ell_{\alpha'}^{(2k)}(x) - \ell_{\alpha''}^{(2k)}(x)| &\leq \frac{1}{a_n^{2k}} |\alpha'_n \ell_{\alpha'}^{(2k)}(L_n^{-1}(x)) - \alpha''_n \ell_{\alpha''}^{(2k)}(L_n^{-1}(x))| \\ &\quad + \frac{1}{a_n^{2k}} |q_n^{(2k)}(\alpha'_n, L_n^{-1}(x)) - q_n^{(2k)}(\alpha''_n, L_n^{-1}(x))|. \\ &\leq \frac{|\alpha'_n|}{a_n^{2k}} |\ell_{\alpha'}^{(2k)}(L_n^{-1}(x)) - \ell_{\alpha''}^{(2k)}(L_n^{-1}(x))| \\ &\quad + \frac{|\alpha'_n - \alpha''_n|}{a_n^{2k}} |\ell_{\alpha''}^{(2k)}(L_n^{-1}(x))| \\ &\quad + \frac{|\alpha'_n - \alpha''_n|}{a_n^{2k}} \left| \frac{\partial}{\partial \alpha_n} q_n^{(2k)}(\xi_n, L_n^{-1}(x)) \right| \end{aligned}$$

for some ξ_n lying between α'_n and α''_n . Using Proposition 4.1, the above inequality implies

$$\|\ell_{\alpha'}^{(2k)} - \ell_{\alpha''}^{(2k)}\| \leq \frac{|\alpha'|_{\infty}}{\mu^{2k}} \|\ell_{\alpha'}^{(2k)} - \ell_{\alpha''}^{(2k)}\| + \frac{|\alpha' - \alpha''|_{\infty}}{\mu^{2k}} (\|\ell_{\alpha''}^{(2k)}\|_{\infty} + M_{2k,2p})$$

which gives (4.10).

The estimates on L^{∞} -error between even order derivatives of the classical Lidstone Interpolation Function and corresponding derivatives of Lidstone FIF for a given data are now given by the following corollary of Theorem 4.2:

Corollary 4.2 For a data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ and scaling factor $\alpha \in \Theta_{2p}$, let ℓ_α be the Lidstone FIF determined by Theorem 3.1 and ϕ be the classical Lidstone Interpolation Function. Then, for $k = 1, 2, \dots, p$,

$$\|\ell_\alpha^{(2k)} - \phi^{(2k)}\|_\infty \leq \frac{|\alpha|_\infty}{\mu^{2k} - |\alpha|_\infty} (\|\phi^{(2k)}\|_\infty + M_{2k,2p}) \quad (4.11)$$

where, $\mu = \min\{a_n : 1 \leq n \leq N\}$ and $M_{2k,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^{p-k} \left(\frac{x_N - x_0}{\pi}\right)^{2l}$.

Proof Inequality (4.11) is an immediate consequence of Inequality (4.10) with $\alpha' = \alpha$ and $\alpha'' = 0$ since $\ell_0 = \phi$.

Using Corollary 4.2, the order of L^∞ -error between even order derivatives of data generating function and the corresponding derivatives of its Lidstone FIF is found in the following theorem:

Theorem 4.4 Let $x_0 < x_1 < \dots < x_N$ be a uniform partition of $[x_0, x_N]$ and $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ be a data generated by a function $g \in C^{2p}[x_0, x_N]$. For $\alpha \in \Theta_{2p}$, let ℓ_α be the Lidstone FIF for the above data determined by Theorem 3.1. Then, for $k = 1, 2, \dots, p$,

$$\|g^{(2k)} - \ell_\alpha^{(2k)}\|_\infty = O(N^{-(2p-2k)}). \quad (4.12)$$

Proof Inequalities (2.3), (4.11) and the triangle inequality give

$$\begin{aligned} \|g^{(2k)} - \ell_\alpha^{(2k)}\|_\infty &\leq 2d_{2p,2k} \|\Delta\|^{2p-2k} \|g^{(2p)}\|_\infty \\ &\quad + \frac{|\alpha|_\infty}{\mu^{2k} - |\alpha|_\infty} [2d_{2p,2k} \|\Delta\|^{2p-2k} \|g^{(2p)}\|_\infty + \|g^{(2k)}\|_\infty + M_{2k,2p}] \end{aligned} \quad (4.13)$$

where $M_{2k,2p} = \frac{2\rho\pi}{3} \sum_{l=0}^{p-k} \left(\frac{x_N - x_0}{\pi}\right)^{2l}$ and $d_{2p,2k}$ is given by (2.4). As $|\alpha|_\infty < \frac{1}{N^{2p}} = \frac{\|\Delta\|^{2p}}{(x_N - x_0)^{2p}}$, there exists $s > 0$ such that $|\alpha|_\infty = \frac{1}{N^{2p+s}}$ and therefore $\frac{|\alpha|_\infty}{\mu^{2k} - |\alpha|_\infty} < \frac{1}{N^{2p+s-2k-1}}$. Consequently, Inequality (4.13) becomes

$$\begin{aligned} \|g^{(2k)} - \ell_\alpha^{(2k)}\|_\infty &\leq 2d_{2p,2k} \|\Delta\|^{2p-2k} \|g^{(2p)}\|_\infty \\ &\quad + \frac{2d_{2p,2k} \|\Delta\|^{2p-2k} \|g^{(2p)}\|_\infty + \|g^{(2k)}\|_\infty + M_{2k,2p}}{N^{2p+s-2k-1}}. \end{aligned}$$

Since $\|\Delta\| = \frac{x_N - x_0}{N}$, the above inequality gives

$$\|g^{(2k)} - \ell_\alpha^{(2k)}\|_\infty \leq \vartheta_k N^{-(2p-2k)} \quad (4.14)$$

where

$$\vartheta_k = \frac{1}{1 - N^{-(2p+s-2k)}} \left[2d_{2p,2k}(x_N - x_0)^{2p-2k} \|g^{(2p)}\|_\infty + \frac{\|g^{(2k)}\|_\infty + M_{2k,2p}}{N^{2p+s-2k}} \right].$$

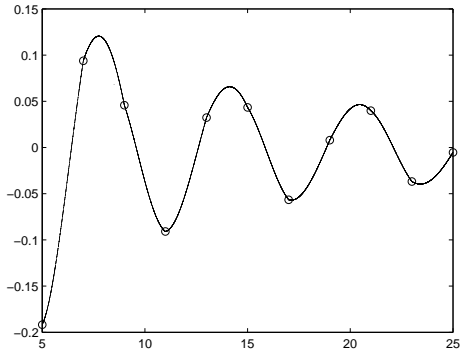
The Estimate (4.12) follows from Inequality (4.14), since it is easily seen that ϑ_k is bounded for $N > 1$.

5 Computational Modeling of Lidstone FIF and its Derivatives

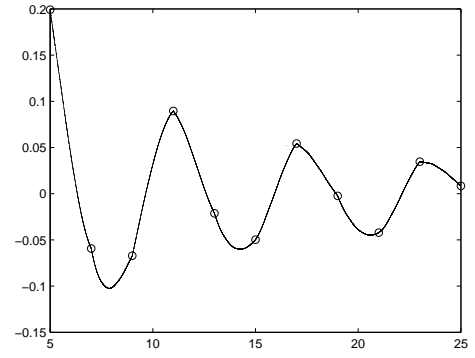
The results obtained in Section 3 are illustrated in this section by computationally constructing a C^4 -Lidstone FIF for a data $\{(x_n, y_{n,k}); n = 0, 1, \dots, 10, k = 0, 1, 2\}$ generated by the function $\frac{\sin x}{x}$ in the interval $[5, 25]$. A generated data along with chosen scaling factors are given in Table 1. The random iteration algorithm with 20000 iterations is used to simulate the C^4 -Lidstone FIF and its derivatives using the construction method developed in Theorem 3.1. The resulting plots of Lidstone FIF and its second and fourth order derivatives are given in Figure 1.

n	x_n	$y_{n,0}$	$y_{n,2}$	$y_{n,4}$	α_n
0	5	-0.1918	0.1991	-0.0508	—
1	7	0.0939	-0.0593	0.1409	0.3162×10^{-4}
2	9	0.0458	-0.0672	-0.0092	0.3981×10^{-4}
3	11	-0.0909	0.0895	-0.0819	0.1585×10^{-4}
4	13	0.0323	-0.0212	0.0523	0.1995×10^{-4}
5	15	0.0434	-0.0497	0.0272	0.2512×10^{-4}
6	17	-0.0566	0.0543	-0.0581	0.3981×10^{-4}
7	19	0.0079	-0.0024	0.0188	0.3802×10^{-4}
8	21	0.0398	-0.0421	0.0337	0.2512×10^{-4}
9	23	-0.0368	0.0346	-0.0400	0.3090×10^{-4}
10	25	-0.0053	0.0084	0.0012	0.3162×10^{-4}

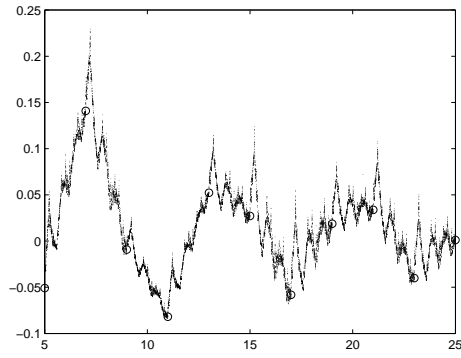
Table 1: A data generated by $\frac{\sin x}{x}$ and chosen scaling factors α_n



(a) c^4 -Lidstone FIF



(b) Second derivative of c^4 -Lidstone FIF



(c) Fourth derivative of c^4 -Lidstone FIF

Figure 1: c^4 -Lidstone FIF for given data and chosen scaling factors (c.f. Table 1)

6 Conclusions

The classical Lidstone Interpolation is extended in the present paper as Lidstone FIF to simulate a given data $\{(x_n, y_{n,2k}); n = 0, 1, \dots, N \text{ and } k = 0, 1, \dots, p\}$ with $N, p \in \mathbb{N}$. The existence of Lidstone FIF is established in our work and a computational method for its construction is developed. The constructed Lidstone FIF is a $C^{2p}[x_0, x_N]$ fractal function ℓ_α satisfying $\ell_\alpha^{(2k)}(x_n) = y_{n,2k}$, $n = 0, 1, \dots, N$, $k = 0, 1, \dots, p$. Our error estimates establish that the order of L^∞ -error in approximation of a data generating function in $C^{2p}[x_0, x_N]$ by Lidstone FIF is of the order N^{-2p} , while L^∞ -error in approximation of $2k$ -order derivative of the data generating function by corresponding order derivative of Lidstone FIF is of the order $N^{-(2p-2k)}$. The results found in the present work are illustrated for computational constructions of a Lidstone FIF and its derivatives with an example of a data generating function.

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